Fermi National Accelerator Laboratory

FERMILAB-Conf-88/24-A
February 1988

Statistical Properties of Strings

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Abstract

The statistical properties of a network of cosmic strings in flat space time are analysed using the microcanonical ensemble. This technique, based on the quantised bosonic string shows that the system is characterised by two distinct phases, corresponding to string densities above and below a "critical" density defined in terms of the string tension. The importance of these results for the cosmic string theory of galaxy formation is then discussed. Finally, it is pointed out why the canonical ensemble is not a good description of strings at high densities.

To appear in the Proceedings of the 3rd University of California Conference on Statistical Mechanics, Davis, California, 27-30 March 1988 (Ed. C. Garrod).

1 Introduction

It is possible that topological defects could have formed following a phase transition in the early universe [1]. Cosmic strings are one dimensional defects with a mass per unit length, μ, determined by the temperature, T_c, of the symmetry breaking. The strings I shall consider were formed at the GUT scale and have a correspondingly large mass per unit length, $\mu \approx T_c^2 \approx 10^{21} kgm^{-1}$ [2]. After formation, the strings evolve in an expanding universe, and one can describe their motion by the action for a fundamental (Nambu) string [3], this approximation being valid for strings whose typical curvature scale is very much greater than their width - this is usually an excellent approximation as the typical scale on the string might be light years whereas the string width is about $10^{-30} m$. However, it is possible for this approximation to break down in regions of high string curvature [4].

Simulations of string formation [5,6,7] proceed by discretising space and then laying down phases of a free Higgs field. Strings form when the Higgs field wraps around the vacuum as one goes around a loop in spacetime. The picture coming from these simulations of string formation is that the network is dominated by one long string that takes up roughly 80% of the energy. The remainding 20% is in the form of closed loops that follow a "scale-invariant" distribution

$$n(R)dR \propto \frac{dR}{R}\frac{1}{R^3} \tag{1}$$

where n(R)dR is the number of loops that have average radius between R and $R + dR$. The evolution of this network in an expanding universe has also been studied [6,7] and whilst quantitative aspects of the simulations may differ slightly, the overall qualitative picture is remarkably similar. I will give an "optimist's" view of the results first and then point out a possible problem that has to be addressed in an analytic framework. The loops of string that formed are under high tension and oscillate, this causes them to radiate gravitationally, eventually disappearing - by themselves they are never a problem and one can show that their contribution to density fluctuations is a constant and moreover is the right constant for galaxies to have formed by today. The long strings cannot radiate themselves away and their only effective energy loss mechanism is chopping off small loops. This has the effect of straightening out the long strings and hence reducing the energy contained in them. If this happens then the scale of the long strings (the length above which they are wiggly) becomes of order the horion size and then their density scales as radiation, $\rho \propto \frac{1}{t^2}$, exactly as required.

2 A Potential Problem

Analytic approaches to the problem of how a string network evolves were started by Kibble [8] and Bennett [9]. Their analyses have taken the form of equations describing the rate of loop production and recombination from and onto the long strings. It is the recombination of loops onto the long string that I ignored in the previous discussion - this process has the effect of not allowing the long strings to straighten out, thus forcing the density in string to decrease slower than radiation. This type of behaviour can be a solution to the equations described above and in this regime the strings would come to dominate the energy density of the universe and hence would automatically rule out cosmic strings as a candidate for a workable theory of galaxy formation. However, it is true to say that these approaches in the end rely on the simulations to fit various unknown parameters, such as the loop production function.

3 A New Approach

In this section I will describe the results of some work that Turok and I did concerning the statistical properties of a network of cosmic strings in flat spacetime [10].

It will turn out that strings have a very peculiar density of states, $\rho(m)$. In fact this is an exponentially increasing function of m,

$$\rho(m) \approx m^{-a} e^{bm} \tag{2}$$

with a and b positive quantities. This is the "Hagedorn" [11] spectrum - first discovered in trying to understand the behaviour of hadrons at very high density. Systems with density of states of this form were first examined by Frautschi[12] and Carlitz[13].

Although I shall be working in the microcanonical ensemble, let me first explain one striking fact about strings using the canonical ensemble. The partition function actually diverges for $\beta > b$:-

$$Z(\beta) = \int dE \rho(E) e^{-\beta E} \approx \int dE e^{-E(b-\beta)} \tag{3}$$

where b is given by (2) and $\beta = \frac{1}{T}$. In other words at temperatures $T > T_c = \frac{1}{b}$ one cannot have strings in thermal equilibrium with a heat bath. The last part of my talk will deal with the canonical ensemble in more detail.

I shall now return to the problem of state counting in the more fundamental microcanonical ensemble. Consider a box of volume V (in flat space) containing strings with total energy between E and $E + dE$, the total number of microstates, Ω, is given by,

$$\begin{aligned} \Omega &= \sum_{n=1}^{\infty} \Omega_n \\ \Omega_n &= \frac{V^n}{n!} \prod_{i=1}^{n} \int_{m_0}^{\infty} \rho(m_i) dm_i \\ &\times \int d^{D+1} p_i \delta(E - \sum_j E_j) \\ &\times \frac{\delta^{D+1}(\sum_j p_j)}{V} \end{aligned} \tag{4}$$

The delta functions constrain the total energy to lie in the desired range and force the total momentum to be zero. The number of loops, n, is allowed to vary and the $n!$ accounts for the indistinguishability of the loops. The quantity D is simply the number of tranverse dimensions. Each loop has the usual centre of mass degrees of freedom plus the internal excitation states described by $\rho(m_i)$. I shall explain why the mass integrals have a lower cutoff, m_0, later on.

The task is now to ask for which n is the summand in (4) maximised. With the usual assumption of ergodicity, the system will end up in configurations corresponding to the maximum value of Ω. In order to calculate $\rho(m_i)$ one has to be able to count states and this can be done by quantising the bosonic string. Cosmic strings have transverse oscillations - these are their physical degrees of freedom. In quantising one has to ensure that it is these degrees of freedom that are being counted. The light cone gauge [14] is ideally suited for this purpose, here one only quantises the tranverse mode oscillators. Some words of explanation are needed here - in general I will not be working in 26 spacetime dimensions, and as such the Lorentz algebra will not close. This is *not* a problem here as I am only interested in counting states of a free theory - interactions are totally neglected apart from their role in allowing the system to equilibrate. Upon solving the constraint equations, it is easy to show that (for closed strings):-

$$m^2 = p^2 = 4\mu\pi(N + \tilde{N}) \tag{5}$$

$$N = \tilde{N} \tag{6}$$

where N ($\tilde{\mathrm{N}}$) is the level number operator for left (right) moving modes. With these two bits of information and a little number theory [15,16,10] one can show for a closed string that,

$$\rho(m)dm \approx m^{-(D+2)} e^{bm} dm \tag{7}$$

with $b = \sqrt{\frac{\pi D}{3\mu}}$. Now I will explain why a lower cutoff in the mass integrals in (4) was put in by hand. I mentioned in the introduction that cosmic strings can only be approximated by the Nambu action if their radius $\gg$ width, this means one can only consider string masses $\gg \mu^{1/2}$ as the string width is given roughly by $\mu^{-1/2}$. This imposes a lower cutoff in the string loop mass. Secondly, the density of states formula (7) was derived using the partition function of Hardy and Ramanujan [16] which is only valid for large N - from (5) this implies large m, also meaning one must impose a lower cutoff in the integrals.

Let me now return to the integrals in (4). Having calculated $\rho(m_i)$ one can now proceed to do them. However, in this short talk I do not have the time (or the inclination!) to enter into any detail concerning their evaluation but I shall refer you to the literature [10]. It turns out that the network has very different behaviour in two density regimes. The critical density $\rho_0 \approx m_0^4 \approx \mu^2$ - this is the density at which the smallest loops have separations of order their radius.

4 High Density results

This corresponds to energies $\gg nm_0$. One can reduce the integrals in (4) down to,

$$\Omega_n \approx \prod_{i=1}^{n} \int_{m_0} dm_i \Theta(E - \sum_j m_j) \\ \times \; e^{bm_i} \sqrt{m_i^{-(D+3)}} \tag{8}$$

This is clearly maximised when $(n-1)$ strings have masses around m_0 and the remainding string takes up the vast majority of the energy $E-(n-1)m_0$. It is trivial to re-express this in terms of energy densities:-

$$\rho - \rho_{inf} = constant \tag{9}$$

where ρ_{inf} is the density in the long string. This formula has been checked

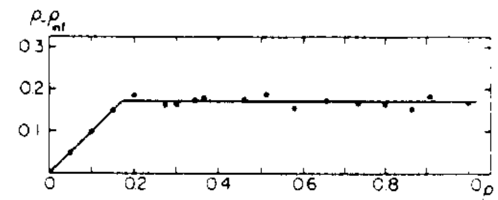

Fig. 1. Results from Sakellariadou and Vilenkin [17], showing the energy density of finite loops as a function of the total string energy density ρ . The density is plotted in units of μ^2 and the transition between high and low density occurs around $\rho = 0.2$.

by a numerical simulation of cosmic string formation in flat spacetime [17], where one can exactly integrate the equations of motion. I would also like to point out that this approach predicts that most of the energy goes into one long string, just as is seen in other simulations [5,6,7].

One can also ask after the 20% of string that is in the form of smaller loops. It is very easy to derive number distributions for these strings by fixing one of them to have a mass M in (4). This then gives the number of microstates for one string to have mass M and the rest to have energy $(E-M)$. Since by assumption all microstates are equally probable, this number must be directly proportional to the probability of one string having mass M, this in turn is proportional to $n(M)$, the number of strings with mass between M and $M+dM$. One can show [10],

$$n(M) \propto \sqrt{M^{-(D+3)}} \tag{10}$$

and using the result [10] that excited string trajectories follow Brownian walks, one has,

$$n(r)dr \propto \frac{dr}{r}\frac{1}{r^{D+1}} \tag{11}$$

which is the "scale-invariant" distribution seen in simulations of string formation.

5 Low Density Results

Here one finds that Ω_n is maximised for $n \approx \frac{E}{m_0}$ and hence all strings are close to their smallest mass. Quantifying this, it can be shown that [10]

$$n(M) \approx e^{-\alpha M/m_0}\sqrt{M^{-(D+3)}} \tag{12}$$

where $\alpha = ln(\frac{\rho_0}{\rho})$ and is bigger than one for low densities. This formula has also been checked against results coming from string simulations at low density in flat spacetime [18].

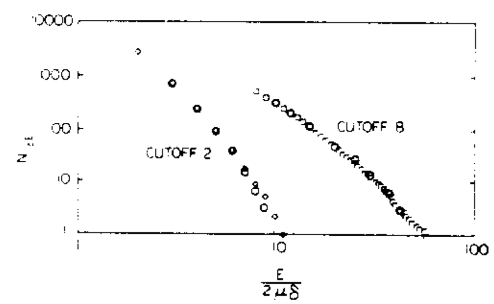

Fig. 2. The squares are the results of Smith and Vilenkin [18] and the circles correspond to equation (12). The amplitude of the distribution and the value of α were fitted using the results for a cutoff of 2 in energy and then these values were used to check agreement with the results for a cutoff of 8.

6 Discussion

The results at low density show that in flat spacetime, the equilibrium distribution of strings is one in which the string grinds itself up into the smallest possible loops. This means that the reconnection probability is very low and hence string domination is extremely unlikely. What I have just said completely neglects the effect of expansion and so may be thought to be too naive. However, let me consider the regime where the length scale on the long string is a decreasing function of the horizon size ($\approx t$). This means the density in long strings is decreasing slower than radiation and string domination will result (this is allowed by the approaches of Kibble [8] and Bennett [9].)

It is well known that once the scale on the network is well inside the horizon the effects of expansion become unimportant and the flat-space results are valid - where the network is *not* dominated by one long string. The only question is then, how quickly does the system reach equilibrium? If one makes the guess that this is of order the string scale then within a time *very much less* than one expansion time, the long string will chop off loops and staighten out until the scale on the string becomes of order the horizon. At this point the density in string will scale as radiation and the network naturally re-enters the "scaling solution." Finally I would like to add that these results have recently been rederived by considering the Higgs field degrees of freedom rather than the quantised bosonic string as I have done here [20].

7 Which Ensemble ?

In the Canonical ensemble one calculates quantities at a given temperature. However, as can be seen from (3) at high temperature, the integrand is *not* a sharply peaked function of E and hence one cannot associate a definite energy with a given temperature. Indeed one can calculate the moments of the energy and they diverge as T approaches $T_c = 1/b$ [10]. In other words the system is characterised by large fluctuations.

However, at low densities (i.e. temperatures well below T_c) one can safely use the canonical ensemble and calculate moments of the energy, specific heats etc.. Moreover, one can show that the specific heat is positive even though there is a finite density of massive modes (see (12)) - contrary to previous claims that c_v is negative for massive string modes [19].

8 Acknowledgements

This work was supported by the SERC of Britain. I would also like to thank Prof. E. Kolb and the Theoretical Astrophysics Group for their hospitality at Fermilab, where it was supported by the DOE and NASA.

References

[1] T.W.B. Kibble, *J. Phys. A9* (1976) 1387.

[2] T.W.B. Kibble, *Phys. Reports 67* (1980) 183;
N. Turok, *"Two Lectures on the Cosmic String Theory of Galaxy Formation" CERN/ESO Winter School on Cosmology and Particle Physics* (1987);
A. Vilenkin, *Phys. Rev. Lett. 46* (1981) 1169, 1496(E), *Phys. Reports 121* (1985) 263;
Ya.B. Zel'dovich, *Mon. Not. Roy. Ast. Soc 192* (1980) 663.

[3] K.B. Nielsen and P. Olesen, *Nucl. Phys. B61* (1973) 45.

[4] K. Maeda and N. Turok, Fermilab preprint *Pub-87/209-A*.

[5] T. Vachaspati and A. Vilenkin, *Phys. Rev. D30* (1984) 2036.

[6] A. Albrecht and N. Turok, *Phys. Rev. Lett. 54* (1985) 1868.

[7] D. Bennett and F. Bouchet, *Phys. Rev. Lett. 60* (1988) 257.

[8] T.W.B. Kibble, *Nucl. Phys. B252* (1985) 227.

[9] D. Bennett, *Phys. Rev. D33* (1986) 872, *D34* (1986) 3592.

[10] D. Mitchell and N. Turok, *Phys. Rev. Lett. 58* (1987) 1577, *Nucl. Phys. B294* (1987) 1138.

[11] R. Hagedorn, *Nuovo Cimento Suppl. 3* (1965) 147.

[12] S. Frautschi, *Phys. Rev. D3* (1971) 2821.

[13] R. Carlitz, *Phys. Rev. D5* (1972) 3231.

[14] P. Goddard, J. Goldstone, C. Rebbi, C. Thorn, *Nucl. Phys. B56* (1973) 109.

[15] K. Huang, S. Weinberg, *Phys. Rev. Lett. 25* (1970) 895.

[16] G. Hardy, S. Ramanujan, *Proc. Lon. Math. Soc. 17* (1918) 895.

[17] M. Sakellariadou and A. Vilenkin, Tufts Preprint *TUTP-87- 14*.

[18] G. Smith and A. Vilenkin, *Phys. Rev. D36* (1987) 990.

[19] M. Bowick, L. Smolin, L. Wijewardhana, *Phys. Rev. Lett. 56* (1986) 424.

[20] E. Copeland, D. Haws, R. Rivers, Fermilab Preprint 1988.